

Brown-Gerver-Ramsey Theorems in Small Dimensions

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Abstract

We consider infinite walks in $\mathbb{N}^k$ with standard unit basis vector steps that avoid t collinear points, and show that these walks exist for $(k, t) \in \{(6, 3), (4, 4), (3, 7)\}$. In particular, our construction for $k = 3$ improves the previous bound 189, obtained by Lidbetter, to 7. Our results also imply the existence of infinite words over small finite alphabets that are weakly abelian squarefree (resp., weakly abelian cubefree, weakly abelian 6th-power-free).

1 Introduction

In 1971 Tom C. Brown observed that every infinite walk on the lattice points of the plane, using only the steps $(0, 1)$ and $(1, 0)$, must contain n collinear points for all $n \geq 2$. He submitted his result to the Advanced Problem section of the *American Mathematical Monthly* [3], and a nice solution was published in 1972 by Peter Montgomery [12].

Of course, it is easy to create such a walk for which every line intersects it at only finitely many points: for example the following walk resembling a parabola, set $P_0 = (0, 0)$, and

$$P_{n+1} = \begin{cases} P_n + (0, 1), & \text{if } P_n = (k^2, k - 1) \text{ for some integer } k; \\ P_n + (1, 0), & \text{otherwise.} \end{cases}$$

Brown's observation was generalized by Gerver and Ramsey to the case where the steps are chosen from an arbitrary fixed finite set $S \subseteq \mathbb{N}^2$ [8]. Gerver and Ramsey also found an upper bound on the walk length (defined as number of steps) guaranteeing the existence of n collinear points, as a function of the size of S and the norm of the largest step. Gerver [7]

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proved a superpolynomial lower bound on the walk length. These upper and lower bounds were recently improved by Korsky [9].

Gerver and Ramsey addressed the same question in $\mathbb{N}^3$. For that version of the problem, they proved there is an infinite walk, using only the unit vector steps $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, with no $5^{11} + 1 = 48828126$ collinear points. (This number was recently reduced to 189 by Lidbetter [11].) They also raised the question of what dimension k is required to avoid three collinear points with a finite set of steps.

Recently this latter question was solved by the first two authors [4]. Their preprint proved the existence of an infinite walk in $\mathbb{Z}^3$ using a set of 16 steps.

In this paper we return to the original question of Brown and restrict our attention to walks in $\mathbb{N}^k$ using only the *standard unit basis vectors*, that is, those of the form

$$\mathbf{e}_{k,i} = (\overbrace{0, \dots, 0}^i, 1, \overbrace{0, \dots, 0}^{k-i-1})$$

for $0 \leq i < k$. By the points of a walk we mean its visited lattice points. The results in [4] immediately show that there is an infinite walk in $\mathbb{N}^{16}$, using only the standard unit basis vectors, with no 3 collinear points [13], but we improve this here.

We now summarize the main results of this paper. All these walks use only the standard unit basis vectors as steps.

- Theorem 3: there is an infinite walk on $\mathbb{N}^6$ with no 3 collinear points. In particular this supersedes the recent paper of Korsky [10].
- Theorem 9: there is an infinite walk on $\mathbb{N}^4$ with no 4 collinear points.
- Theorem 10: there is an infinite walk on $\mathbb{N}^3$ with no 7 collinear points.

The proofs are presented in Sections 3 and 4. Readers less interested in combinatorics of words (equivalence explained in Section 2), can skip Corollary 5 in Section 3 and Section 5.

We do not know if these results can be improved for unit basis vector steps. For example, we cannot currently rule out an infinite walk on $\mathbb{N}^3$ with no 4 collinear points. We have constructed a finite walk of length 32768 with this property. However, as Brown observed [2], the longest walk on $\mathbb{N}^3$ with no 3 collinear points is of length 7.

We also cannot rule out an infinite walk in $\mathbb{N}^4$ with no 3 collinear points. We have constructed a finite walk of length 5500 with this property.

Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$, and word and vertex indices start at zero.

Our constructions use p -adic valuations for the primes $p = 2$ and $p = 3$. For a nonzero integer n , we let $\nu_p(n)$ be the exponent of the highest power of p dividing n . This is extended to nonzero rational numbers m/n by $\nu_p(m/n) = \nu_p(m) - \nu_p(n)$.

2 Weak abelian powers

Our results have a natural interpretation in combinatorics on words. Let $\Sigma_k = \{0, 1, \dots, k-1\}$ be an alphabet of cardinality k . For a word w , let $|w|_a$ be the number of occurrences of the letter a in w . For $w \in \Sigma_k^*$ define its Parikh vector $\psi(w)$ to be $(|w|_0, |w|_1, \dots, |w|_{k-1})$. We say that a word w is a *weak abelian t 'th power* if $w = x_1 x_2 \cdots x_t$, where each word x_i is nonempty and

$$\frac{\psi(x_1)}{|x_1|} = \frac{\psi(x_2)}{|x_2|} = \dots = \frac{\psi(x_t)}{|x_t|}.$$

We say a word w *avoids* weak abelian t 'th powers if no nonempty factor is a weak abelian t 'th power. See, for example, [1, 6].

Proposition 1. *Let $w = w_0 w_1 \cdots w_{n-1} \in \Sigma_k^n$. Define*

$$P_i = \sum_{0 \leq j < i} \mathbf{e}_{k, w_j}$$

for $0 \leq i \leq n$. Then the walk $P_0, \dots, P_n$ contains no $t+1$ collinear points if and only if w avoids weak abelian t 'th powers.

Proof. If there are $t+1$ collinear points, their consecutive displacements are positive multiples of a common vector; division by their coordinate sums gives the same normalized Parikh vector for all t intervening nonempty blocks. Conversely, such t blocks give $t+1$ collinear boundary points. $\square$

Thus our results can be restated as

- There is an infinite word over Σ_6 with no weak abelian squares.
- There is an infinite word over Σ_4 with no weak abelian cubes.
- There is an infinite word over Σ_3 with no weak abelian 6th powers.

We also produce explicit constructions for these three words, as the image of a fixed point of a morphism.

Remark 2. Given a finite word $w = w_0 w_1 \cdots w_{n-1}$, we can determine the smallest exponent $p \geq 2$ such that w avoids weak abelian p 'th powers in $O(n^2 \log n)$ time and $O(n)$ space. For each $0 \leq j < n$, compute

$$\frac{\psi(w_j w_{j+1} \cdots w_{\ell-1})}{\ell - j} \quad (j < \ell \leq n),$$

sort these vectors lexicographically, separately for each j , and find the maximum multiplicity M of an identical vector. Then $p = M + 1$. This can be improved to $O(n^2)$ time and $O(n)$ space using radix sort with the appropriate buckets.

3 Dimension 6

Theorem 3. *There is an infinite walk in $\mathbb{N}^6$, using only the standard unit basis vectors as steps, with no 3 collinear points.*

We begin with a walk on the Gaussian integers, adapted from the construction in [4]. Let $i = \sqrt{-1}$, and define $\sigma_n \in \{0, 1, 2, 3\}$ recursively by $\sigma_0 = 0$ and

$$\sigma_{2n} \equiv -\sigma_n \pmod{4}, \quad \sigma_{2n+1} \equiv 1 - \sigma_n \pmod{4}. \quad (1)$$

For $n \geq 0$, define

$$u_n = i^{\sigma_n}, \quad z_n = \sum_{0 \leq r < n} u_r \in \mathbb{Z}[i]. \quad (2)$$

Thus (z_n) is a nearest-neighbor walk in the Gaussian lattice. For $\varepsilon \in \{0, 1\}$, Eq. (1) gives

$$u_{2n+\varepsilon} = i^\varepsilon \overline{u_n}, \quad z_{2n+\varepsilon} = (1+i) \overline{z_n} + \varepsilon \overline{u_n}, \quad (3)$$

where $\overline{x}$ is the complex conjugate. Indeed, the second identity follows by pairing $u_{2r} + u_{2r+1} = (1+i)\overline{u_r}$.

Lemma 4. *If $0 \leq m < n$ and $\sigma_m = \sigma_n$, then $z_m \neq z_n$ and*

$$\nu_2(|z_n - z_m|^2) = \nu_2(n - m). \quad (4)$$

Proof. Let $t = \nu_2(n - m)$ and set $(m_0, n_0) = (m, n)$. We recursively define pairs (m_j, n_j) satisfying $n_j - m_j = (n - m)/2^j$ for $0 \leq j \leq t$. Suppose that $j < t$. Then $n_j - m_j$ is even, so m_j and n_j have the same parity. Thus, for a unique $\varepsilon_j \in \{0, 1\}$, we may write

$$m_j = 2m_{j+1} + \varepsilon_j, \quad n_j = 2n_{j+1} + \varepsilon_j.$$

The two maps

$$r \mapsto -r, \quad r \mapsto 1 - r$$

are injective on $\mathbb{Z}/4\mathbb{Z}$. Hence the recursion (1), together with $\sigma_{m_j} = \sigma_{n_j}$, implies $\sigma_{m_{j+1}} = \sigma_{n_{j+1}}$. In particular, $u_{m_{j+1}} = u_{n_{j+1}}$. Applying Eq. (3) with $n = m_{j+1}$ and $n = n_{j+1}$, we get

$$z_{n_j} - z_{m_j} = (1+i)(\overline{z_{n_{j+1}}} - \overline{z_{m_{j+1}}}).$$

Consequently,

$$|z_{n_j} - z_{m_j}|^2 = 2|z_{n_{j+1}} - z_{m_{j+1}}|^2.$$

Iterating for $j = 0, \dots, t-1$ yields

$$|z_n - z_m|^2 = 2^t |z_{n_t} - z_{m_t}|^2.$$

By construction, $n_t - m_t$ is odd. Moreover,

$$z_{n_t} - z_{m_t} = \sum_{r=m_t}^{n_t-1} u_r$$

is a sum of an odd number of elements of $\{1, i, -1, -i\}$. Write this sum as $x + iy$, where $x + y$ is odd. Thus exactly one of x and y is odd, so

$$|z_{n_t} - z_{m_t}|^2 = x^2 + y^2$$

is odd and, in particular, nonzero. It follows that $z_n - z_m \neq 0$ and $\nu_2(|z_n - z_m|^2) = t = \nu_2(n - m)$. $\square$

Proof of Theorem 3. Let $\delta_n = (\sigma_{n+1} - \sigma_n) \bmod 4$. From Eq. (1),

$$\delta_{2n} = 1, \quad \delta_{2n+1} \equiv -1 - \delta_n \pmod{4}. \quad (5)$$

Since $\delta_0 = 1$, induction gives $\delta_n \in \{1, 2\}$ for every n . Consequently, the only possible ordered pairs (σ_n, σ_{n+1}) are

$$(0, 1), (1, 2), (2, 3), (3, 0), (0, 2), (1, 3), (2, 0), (3, 1). \quad (6)$$

We now define two functions on $\{0, 1, 2, 3\}$ by

$$(d_0, d_1, d_2, d_3) = (0, 1, i, 1 + i), \quad (c_0, c_1, c_2, c_3) = (0, 0, 0, 1 - i),$$

and define the auxiliary points

$$R_n = (d_{\sigma_n}, z_n + c_{\sigma_n}, n).$$

If $(\sigma_n, \sigma_{n+1}) = (r, s)$, then

$$R_{n+1} - R_n = (d_s - d_r, i^r + c_s - c_r, 1). \quad (7)$$

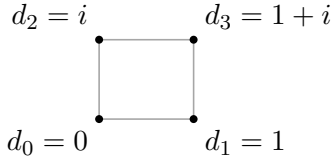
The values d_r place the four possible values of σ_n at the extreme points of a square, while the offsets c_r make the possible increments of R_n collapse from eight ordered pairs to the six vectors shown in Figure 1.

We claim that no three points R_n are collinear. Suppose otherwise, and choose $a < b < c$ such that R_a, R_b, R_c are collinear. Write $p = b - a$ and $q = c - b$. Since the last coordinate of R_n is n , the affine parameter is fixed:

$$\frac{R_b - R_a}{p} = \frac{R_c - R_b}{q} = \frac{R_c - R_a}{p + q}. \quad (8)$$

In the first complex coordinate this says

$$d_{\sigma_b} = \frac{q}{p + q} d_{\sigma_a} + \frac{p}{p + q} d_{\sigma_c}. \quad (9)$$



(σ_n, σ_{n+1})	$R_{n+1} - R_n$
01	$(1, 1, 1)$
12	$(-1 + i, i, 1)$
23	$(1, -i, 1)$
30	$(-1 - i, -1, 1)$
02, 13	$(i, 1, 1)$
20, 31	$(-i, -1, 1)$

Figure 1: Collapse to six steps.

The four numbers d_0, d_1, d_2, d_3 are the vertices of a square. Since all four are extreme points of their convex hull, the strict convex combination in (9) forces

$$\sigma_a = \sigma_b = \sigma_c. \quad (10)$$

The correction terms c_{σ_n} now cancel from the second complex coordinates in (8). Hence

$$\frac{z_b - z_a}{p} = \frac{z_c - z_b}{q} = \frac{z_c - z_a}{p + q}. \quad (11)$$

By Lemma 4, for example, we get the square-modulus valuations

$$\nu_2 \left(\left| \frac{z_b - z_a}{p} \right|^2 \right) = \nu_2(|z_b - z_a|^2) - 2\nu_2(p) = -\nu_2(p).$$

Thus the three nonzero fractions in (11) have squared-modulus valuations

$$-\nu_2(p), \quad -\nu_2(q), \quad -\nu_2(p + q), \quad (12)$$

respectively. Equality in (11) would therefore give $\nu_2(p) = \nu_2(q) = \nu_2(p + q)$. This is impossible: if p and q have the same 2-adic valuation, then $p + q$ has a larger one. Thus no three points of the auxiliary walk (R_n) are collinear.

Finally, list the six possible increments of R_n in Figure 1 as $v_1, \dots, v_6$. Starting from $P_0 = 0$, replace every occurrence of v_j by the standard basis vector $\mathbf{e}_{6,j-1} \in \mathbb{N}^6$. The real-linear map

$$T : \mathbb{R}^6 \longrightarrow \mathbb{C}^2 \times \mathbb{R}, \quad T(\mathbf{e}_{j-1}) = v_j,$$

satisfies $T(P_n) = R_n$ for every n . Moreover, the sum of the coordinates of P_n is n , so the points P_n are distinct. A collinear triple among the P_n would map to three distinct collinear points among the R_n , which we have just proved impossible. This completes the construction. $\square$

Corollary 5. *The steps of the walk are given by the unit basis vectors $(\mathbf{e}_{6,q_i})_{i \geq 0}$, where q_i is the image under τ of the fixed point of the morphism γ , defined as follows:*

j	$\gamma(j)$	$\tau(j)$
0	01	0
1	20	1
2	34	2
3	25	3
4	62	4
5	03	1
6	67	5
7	36	4

Thus

$$\tau(\gamma^\omega(0)) = 01203401215201203403543401203401 \dots$$

is weakly abelian squarefree.

Proof. Let $\mathbf{t} = t_0 t_1 t_2 \dots = \gamma^\omega(0)$. Associate each letter with an ordered pair by

j	0	1	2	3	4	5	6	7
$\pi(j)$	(0, 1)	(1, 3)	(3, 0)	(1, 2)	(2, 0)	(0, 2)	(2, 3)	(3, 1).

We claim that

$$\pi(t_n) = (\sigma_n, \sigma_{n+1}) \quad (n \geq 0).$$

Indeed, if $\pi(j) = (r, s)$ and $\gamma(j) = j_0 j_1$, direct inspection of the table gives

$$\pi(j_0) = (-r, 1 - r), \quad \pi(j_1) = (1 - r, -s),$$

where all entries are reduced modulo 4. On the other hand, the recurrence (1) gives

$$\begin{aligned} (\sigma_{2n}, \sigma_{2n+1}) &= (-\sigma_n, 1 - \sigma_n), \\ (\sigma_{2n+1}, \sigma_{2n+2}) &= (1 - \sigma_n, -\sigma_{n+1}), \end{aligned}$$

again modulo 4. Since $\pi(t_0) = \pi(0) = (0, 1) = (\sigma_0, \sigma_1)$ and $t_{2n} t_{2n+1} = \gamma(t_n)$, the claim follows by induction.

Now, if $\pi(t_n) = (r, s)$, the corresponding increment of the auxiliary walk is

$$R_{n+1} - R_n = (d_s - d_r, i^r + c_s - c_r, 1).$$

By Figure 1, the pairs associated with letters 1 and 5, namely (1, 3) and (0, 2), give the same increment. Similarly, the pairs associated with letters 4 and 7, namely (2, 0) and (3, 1), give the same increment. These are the only identifications among the eight pairs. They are exactly the identifications made by τ , since

$$\tau(1) = \tau(5) = 1, \quad \tau(4) = \tau(7) = 4,$$

while the remaining four letters receive the distinct labels 0, 2, 3, 5.

Thus $q_n = \tau(t_n)$ labels precisely the increment $R_{n+1} - R_n$. Replacing these six increment types by $\mathbf{e}_{6,0}, \dots, \mathbf{e}_{6,5}$ therefore gives the walk constructed in the proof of Theorem 3, up to a permutation of coordinates. This walk has no three collinear points, so its step word $\tau(\gamma^\omega(0))$ is weakly abelian squarefree. $\square$

4 Dimensions 4 and 3

The constructions here are similar to those of the previous section, except that we now use base-3 representations and the field norm

$$N_{\mathbb{Q}(\sqrt{3})/\mathbb{Q}}(X + Y\sqrt{3}) = X^2 - 3Y^2.$$

Write $n = \sum_{j \geq 0} f_j 3^j$ in base 3 and set

$$a_n = (-1)^{f_1+f_3+\dots}, \quad b_n = (-1)^{f_0+f_2+\dots}, \quad R_n = (X_n, Y_n, n),$$

where $X_n = \sum_{j < n} a_j$ and $Y_n = \sum_{j < n} b_j$. Appending a ternary digit gives, for $r = 0, 1, 2$,

$$a_{3n+r} = b_n, \quad b_{3n+r} = (-1)^r a_n, \quad (13)$$

and therefore

$$X_{3n+r} = 3Y_n + r b_n, \quad Y_{3n+r} = X_n + \varepsilon_r a_n, \quad (\varepsilon_0, \varepsilon_1, \varepsilon_2) = (0, 1, 0). \quad (14)$$

Now we state and prove an analogue of Lemma 4.

Lemma 6. *If $0 \leq m < n$ and $(a_m, b_m) = (a_n, b_n)$, then $(X_n - X_m)^2 - 3(Y_n - Y_m)^2$ is nonzero and*

$$\nu_3((X_n - X_m)^2 - 3(Y_n - Y_m)^2) = \nu_3(n - m). \quad (15)$$

Proof. Let $t = \nu_3(n - m)$ and set $(m_0, n_0) = (m, n)$. We recursively define pairs (m_j, n_j) for $0 \leq j \leq t$. Suppose that $j < t$. Then $n_j - m_j = (n - m)/3^j$ is divisible by 3, so m_j and n_j have the same remainder modulo 3. Thus, for a unique $r_j \in \{0, 1, 2\}$, we may write

$$m_j = 3m_{j+1} + r_j, \quad n_j = 3n_{j+1} + r_j.$$

For each $r \in \{0, 1, 2\}$, the map

$$(a, b) \mapsto (b, (-1)^r a)$$

is injective on $\{1, -1\}^2$. Hence the recursion (13), together with $(a_{m_j}, b_{m_j}) = (a_{n_j}, b_{n_j})$, implies

$$(a_{m_{j+1}}, b_{m_{j+1}}) = (a_{n_{j+1}}, b_{n_{j+1}}).$$

Applying Eq. (14) with $n = m_{j+1}$ and $n = n_{j+1}$, in both cases with $r = r_j$, and subtracting, the correction terms cancel, giving

$$X_{n_j} - X_{m_j} = 3(Y_{n_{j+1}} - Y_{m_{j+1}}), \quad Y_{n_j} - Y_{m_j} = X_{n_{j+1}} - X_{m_{j+1}}.$$

For $0 \leq j \leq t$, put

$$D_j = (X_{n_j} - X_{m_j})^2 - 3(Y_{n_j} - Y_{m_j})^2.$$

The preceding identities give $D_j = -3D_{j+1}$ for $j < t$. Iterating yields

$$D_0 = (-3)^t D_t.$$

By construction, $n_t - m_t$ is not divisible by 3. Write $m_t = 3u + r$ and $n_t = 3v + s$, where $r, s \in \{0, 1, 2\}$ and $r \neq s$. Since $a_{m_t} = a_{n_t}$, Eq. (13) gives $b_u = b_v$. It follows from (14) that

$$X_{n_t} - X_{m_t} \equiv sb_v - rb_u = (s - r)b_u \not\equiv 0 \pmod{3}.$$

Consequently,

$$D_t \equiv (X_{n_t} - X_{m_t})^2 \equiv 1 \pmod{3}.$$

Thus D_t is nonzero and not divisible by 3. Since $D_0 = (-3)^t D_t$, we conclude that $D_0 \neq 0$ and

$$\nu_3((X_n - X_m)^2 - 3(Y_n - Y_m)^2) = \nu_3(D_0) = t = \nu_3(n - m).$$

□

Lemma 7. *Let $E = \{n \geq 0 : (a_n, b_n) = (1, 1)\}$. No four points R_n with $n \in E$ are collinear.*

Proof. Suppose $R_{n_0}, \dots, R_{n_3}$, with $n_0 < \dots < n_3$ in E , lie on one line. There are rational slopes α, β such that every chord has the form

$$(X_{n_j} - X_{n_i}, Y_{n_j} - Y_{n_i}) = (n_j - n_i)(\alpha, \beta).$$

Put $c = \alpha^2 - 3\beta^2$. Lemma 6 implies that $c \neq 0$ and, for every $i < j$, gives

$$\nu_3(n_j - n_i) = \nu_3(c) + 2\nu_3(n_j - n_i)$$

by Eq. (15). Hence all six differences have one common valuation $e = -\nu_3(c) \geq 0$. The four integers

$$\frac{n_i - n_0}{3^e} \quad (0 \leq i \leq 3)$$

would therefore have pairwise distinct residues modulo 3, which is impossible. □

Enumerate E as

$$0 = N_0 < N_1 < N_2 < \dots.$$

Proposition 8. *For every $k \geq 0$,*

$$N_{k+1} - N_k \in \{2, 4, 6, 8\}.$$

More precisely, if

$$r = \frac{N_{k+1} - N_k}{2},$$

then

$$\frac{1}{2}(R_{N_{k+1}} - R_{N_k}) = d_r, \tag{16}$$

where d_r is defined by the following table:

$$\begin{array}{c|cccc} r & 1 & 2 & 3 & 4 \\ \hline d_r & (1, 0, 1) & (-1, 1, 2) & (0, -1, 3) & (-2, 0, 4) \end{array}. \tag{17}$$

We present two proofs for this proposition, so readers can choose the proof best suited to their background.

First proof. If $n = 9q + 3u + v$, with $u, v \in \{0, 1, 2\}$, then by (13) (applied twice)

$$(a_n, b_n) = ((-1)^u a_q, (-1)^v b_q).$$

The residues modulo 9 giving state $(1, 1)$ given (a_q, b_q) are respectively

$$\begin{array}{c|cccc} (a_q, b_q) & (1, 1) & (1, -1) & (-1, 1) & (-1, -1) \\ \hline n - 9q & 0, 2, 6, 8 & 1, 7 & 3, 5 & 4 \end{array}.$$

Notice that $(\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}, \cdot)$, where $\cdot$ is the entrywise multiplication forms the Klein-four group, i.e., every element is its inverse and so the table can also be read reversely; starting from $(a_0, b_0) = (1, 1)$ one obtains the latter (a_{n-9q}, b_{n-9q}) .

When q is increased by one, a ternary carry changes the parity of exactly one of the two alternating digit sums (notice that $n = \sum_{j \geq 0} f_j 3^j \equiv \sum_{j \geq 0} f_j \pmod{2}$). Noticing that the corresponding residue $n - 9q$ changes parity as well, reading consecutive entries in the above table therefore shows that $N_{k+1} - N_k \in \{2, 4, 6, 8\}$.

In each of the cases, the sum of the (a_j, b_j) depends solely on the difference and one can check (17).

For this calculation, one sums the corresponding (a_q, b_q) , multiplied with the base state (to correspond with $(1, 1)$). For example, if one has $N_i \equiv 5 \pmod{9}$ and $N_{i+1} \equiv 4 \pmod{9}$, we have

$$\begin{aligned} & (-1, 1) \cdot ((-1, 1) + (1, 1) + (1, -1) + (1, 1)) + (-1, -1) \cdot ((1, 1) + (1, -1) + (1, 1) + (-1, 1)) \\ &= (-1, 1) \cdot (2, 2) + (-1, -1) \cdot (2, 2) = (-4, 0), \text{ which is twice } (-2, 0). \end{aligned}$$

The thirteen cases, computed as above, can be summarized as follows

$N_{k+1} - N_k$	$(N_k \pmod{9}) \longrightarrow (N_{k+1} \pmod{9})$	$\frac{1}{2}(R_{N_{k+1}} - R_{N_k})$
2	$0 \rightarrow 2, 3 \rightarrow 5, 6 \rightarrow 8, 7 \rightarrow 0, 8 \rightarrow 1$	$(1, 0, 1) = d_1$
4	$2 \rightarrow 6, 5 \rightarrow 0, 8 \rightarrow 3$	$(-1, 1, 2) = d_2$
6	$1 \rightarrow 7, 4 \rightarrow 1, 7 \rightarrow 4$	$(0, -1, 3) = d_3$
8	$4 \rightarrow 3, 5 \rightarrow 4$	$(-2, 0, 4) = d_4$

Table 1: The thirteen possible transitions between consecutive elements of E . An arrow $r \rightarrow s$ records $N_k \equiv r \pmod{9}$ and $N_{k+1} \equiv s \pmod{9}$.

□

Second proof. Set

$$A = (1, 1), \quad B = (1, -1), \quad C = (-1, 1), \quad D = (-1, -1).$$

The recurrence

$$a_{3n+s} = b_n, \quad b_{3n+s} = (-1)^s a_n \quad (s \in \{0, 1, 2\})$$

induces the three-uniform morphism

$$\mu(A) = ABA, \quad \mu(B) = CDC, \quad \mu(C) = BAB, \quad \mu(D) = DCD. \quad (18)$$

Thus the word

$$\mathbf{x} = ((a_n, b_n))_{n \geq 0},$$

written over the alphabet $\{A, B, C, D\}$, is the fixed point $\mu^\omega(A)$. The occurrences of A in $\mathbf{x}$ are precisely the positions $N_0, N_1, N_2, \dots$

Define morphisms φ and η by

j	$\varphi(j)$	$\eta(j)$	$ \eta(j) /2$
0	01	AB	1
1	0020	$ACDC$	2
2	034	$ABDCDB$	3
3	00224	$ACDCDCDB$	4
4	03220	$ABDCDCDC$	4.

(19)

Let $\mathbf{y} = \varphi^\omega(0)$. Direct substitution gives

$$\begin{aligned} \mu(\eta(0)) &= \eta(0)\eta(1), \\ \mu(\eta(1)) &= \eta(0)\eta(0)\eta(2)\eta(0), \\ \mu(\eta(2)) &= \eta(0)\eta(3)\eta(4), \\ \mu(\eta(3)) &= \eta(0)\eta(0)\eta(2)\eta(2)\eta(4), \\ \mu(\eta(4)) &= \eta(0)\eta(3)\eta(2)\eta(2)\eta(0). \end{aligned}$$

Equivalently,

$$\mu \circ \eta = \eta \circ \varphi.$$

Since $\varphi(\mathbf{y}) = \mathbf{y}$, it follows that

$$\mu(\eta(\mathbf{y})) = \eta(\varphi(\mathbf{y})) = \eta(\mathbf{y}).$$

The word $\eta(\mathbf{y})$ begins with A , and μ has a unique fixed point beginning with A . Hence

$$\eta(\mathbf{y}) = \mathbf{x}.$$

Every word $\eta(j)$ begins with A and contains no other occurrence of A . Therefore the boundaries of the consecutive η -blocks in $\eta(\mathbf{y}) = \mathbf{x}$ are exactly the positions N_k . In particular, for every $k \geq 0$,

$$\mathbf{x}[N_k \dots N_{k+1} - 1] = \eta(y_k), \quad N_{k+1} - N_k = |\eta(y_k)|. \quad (20)$$

The length column of (19) now gives $N_{k+1} - N_k \in \{2, 4, 6, 8\}$.

It remains to compute the corresponding displacement. Regard each letter $L \in \{A, B, C, D\}$ as its associated vector in $\{\pm 1\}^2$. From the definition of R_n and (20),

$$R_{N_{k+1}} - R_{N_k} = \left(\sum_{L \text{ in } \eta(y_k)} L, |\eta(y_k)| \right),$$

where letters are counted with multiplicity. The five possible blocks give

$$\begin{array}{c|c|c} j & \sum_{L \text{ in } \eta(j)} L & \frac{1}{2} \left(\sum_{L \text{ in } \eta(j)} L, |\eta(j)| \right) \\ \hline 0 & (2, 0) & (1, 0, 1) = d_1 \\ 1 & (-2, 2) & (-1, 1, 2) = d_2 \\ 2 & (0, -2) & (0, -1, 3) = d_3 \\ 3 & (-4, 0) & (-2, 0, 4) = d_4 \\ 4 & (-4, 0) & (-2, 0, 4) = d_4. \end{array} \quad (21)$$

Together with (20), this proves (16). $\square$

Theorem 9. *There is an infinite standard-basis walk in $\mathbb{N}^4$ with no 4 collinear points.*

Proof. At the k th step use $\mathbf{e}_{4,r-1}$, where $r = (N_{k+1} - N_k)/2$. The linear map sending $\mathbf{e}_{4,r-1}$ to d_r sends the k th walk vertex to $R_{N_k}/2$. Thus four collinear walk points would have four collinear images; the images are distinct because their third coordinates are the distinct numbers $N_k/2$. Lemma 7 gives the contradiction. $\square$

Theorem 10. *There is an infinite standard-basis walk in $\mathbb{N}^3$ with no seven collinear points.*

Proof. The vectors d_1, d_2, d_3 in (17) are independent (determinant 6), and $d_1 + d_4 = d_2 + d_3$. Hence an invertible linear map may send

$$d_1 \mapsto \mathbf{e}_{3,0}, \quad d_2 \mapsto \mathbf{e}_{3,1}, \quad d_3 \mapsto \mathbf{e}_{3,0} + \mathbf{e}_{3,2},$$

and then necessarily $d_4 \mapsto \mathbf{e}_{3,1} + \mathbf{e}_{3,2}$. The images S_k of $R_{N_k}/2$ therefore form a walk with these four macro-steps, and Lemma 7 says that no four S_k are collinear.

Subdivide $\mathbf{e}_{3,0} + \mathbf{e}_{3,2}$ as $\mathbf{e}_{3,2}, \mathbf{e}_{3,0}$ and $\mathbf{e}_{3,1} + \mathbf{e}_{3,2}$ as $\mathbf{e}_{3,2}, \mathbf{e}_{3,1}$. Every vertex of the resulting unit-step walk lies in

$$\mathcal{S} \cup (\mathcal{S} + \mathbf{e}_{3,2}), \quad \mathcal{S} = \{S_k : k \geq 0\}.$$

Each line meets either translate in at most three points, hence contains at most six points altogether. $\square$

Remark 11. The bound six is attained by this construction. Its vertices at times 64, 70, 82, 88, 100, 106 are

$$(38, 13, 13), (41, 15, 14), (47, 19, 16), (50, 21, 17), (56, 25, 19), (59, 27, 20),$$

and lie on a line of primitive direction $(3, 2, 1)$. Thus Theorem 10 cannot be strengthened to “at most five” by merely sharpening its last estimate.

Also in Theorem 9 the “at most three” is sharp with the given construction, since the second, third and fourth point $P_2 = (1, 1, 0, 0)$, $P_3 = (2, 1, 0, 0)$ and $P_4 = (3, 1, 0, 0)$ are collinear.

5 A common morphism for Theorems 9 and 10

Use the same φ and $\mathbf{y}$ as above and define

j	$\varphi(j)$	$\tau_4(j)$	$\tau_3(j)$
0	01	0	0
1	0020	1	1
2	034	2	20
3	00224	3	21
4	03220	3	21

(22)

Here τ_4 is a letter-to-letter coding; τ_3 is a non-erasing morphism, and 20 and 21 denote two-letter words.

Corollary 12. *The word $\mathbf{w}_4 = \tau_4(\varphi^\omega(0))$ is the step word of the four-dimensional walk in Theorem 9, using zero-based direction labels. In particular, it is weakly abelian cubefree.*

Proof. Equation (20), together with the last column of (19), gives

$$(\mathbf{w}_4)_k = \frac{N_{k+1} - N_k}{2} - 1.$$

Thus $\mathbf{w}_4$ specifies precisely the steps of Theorem 9, with d_r represented by $\mathbf{e}_{4,r-1}$. Equivalently, the linear map $\mathbf{e}_{4,r-1} \mapsto d_r$ sends the k th walk vertex to $R_{N_k}/2$. Lemma 7 rules out four collinear images; these images are distinct because their third coordinates are $N_k/2$. Hence the walk has no four collinear points, and Proposition 1 proves the assertion about weak abelian cubes. $\square$

Corollary 13. *The word $\mathbf{w}_3 = \tau_3(\varphi^\omega(0))$ is the step word of the three-dimensional walk in Theorem 10. In particular, it avoids weak abelian sixth powers.*

Proof. Theorem 10 replaces the return steps d_1, d_2, d_3, d_4 by the unit-step words 0, 1, 20, 21, respectively. By Eq. (21), the return letters 0, 1, 2, 3, 4 therefore give exactly the images 0, 1, 20, 21, 21 defining τ_3 . Consequently $\mathbf{w}_3 = \tau_3(\mathbf{y})$ specifies the stated construction.

For completeness, let S_k be the image of $R_{N_k}/2$ under the invertible linear map in Theorem 10. Lemma 7 implies that each line contains at most three points of $\mathcal{S} = \{S_k : k \geq 0\}$. Every extra vertex introduced by a two-step image is $S_k + \mathbf{e}_{3,2}$. Thus all points lie in $\mathcal{S} \cup (\mathcal{S} + \mathbf{e}_{3,2})$, and every line contains at most six points. Proposition 1 then rules out weak abelian sixth powers. $\square$

Alternatively we can define the word in Theorem 10 as follows:

L	$\theta(L)$	$\pi(L)$
a	ab	0
b	$aacda$	1
c	a	2
d	$efgh$	0
e	a	2
f	$acdcdgh$	1
g	a	2
h	$efcdcdca$	1

$$\mathbf{w}_3 = \pi(\theta^\omega(a)). \tag{23}$$

To verify this, define

$$H(0) = a, \quad H(1) = b, \quad H(2) = cd, \quad H(3) = ef, \quad H(4) = gh.$$

The table satisfies $\theta \circ H = H \circ \varphi$ and $\pi \circ H = \tau_3$. It follows that $H(\mathbf{y})$ is a fixed point of θ starting with a , hence $H(\mathbf{y}) = \theta^\omega(a)$, and therefore

$$\pi(\theta^\omega(a)) = \pi(H(\mathbf{y})) = \tau_3(\mathbf{y}) = \mathbf{w}_3.$$

The limit exists because $\theta(a) = ab$ and the morphism is non-erasing, with $|\theta^n(a)|$ tending to infinity.

The construction of φ is based on return words. For a general statement and proof, see Durand [5, Proposition 19]: the derived sequence at a nonempty prefix of the fixed point of a primitive morphism is itself fixed by a primitive morphism. The identity $\mu \circ \eta = \eta \circ \varphi$ above is a direct certificate in this instance. Proposition 9 of the same paper gives the general conversion from a non-erasing morphic image of a primitive fixed point to a coding of a primitive fixed point; the eight-letter presentation above makes that conversion explicit here.

6 Final words

As we mentioned, we do not know if the result of Theorem 3 could be improved to $\mathbb{N}^5$ or $\mathbb{N}^4$. The third author found a possible construction in $\mathbb{N}^5$ [13], but we have not been able to prove that it works. Namely, consider the cyclic morphism $\alpha(i)$ defined by adding i modulo 5 to every letter of $\alpha(0) = 01213101314310$. Then $\alpha^\omega(0)$ appears to avoid weak abelian squares, at least as far as we have been able to check (600,000 terms), and if correct, this would give a walk in $\mathbb{N}^5$ with no 3 collinear points, by the translation described in Section 2.

A website devoted to explaining the results of this paper is available at

<https://basis-walk.q5m.ai/> ,

and a github repository with code is

<https://github.com/ekalvi/basis-walk> .

Declaration of AI usage

Initial proofs for the results in this paper were obtained with GPT-6 Astra. Everything was checked and rewritten by the authors.

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